Exercise Sheet 2: Introduction to Lusternik-Schnirelmann Category

Tyrone Cutler

May 1, 2020

1 Lusternik-Schnirelmann Category

The invariant we will discuss in these exercises was first introduced by the two Soviet mathematicians L. Lusternik and L. Schnirelmann in the 1930's [3]. Their interest was in finding bounds for the minimum number of critical points of any smooth function defined on a given manifold M. While one might imagine that such a quantity would necessarily depend explicitly on the smooth structure of M, Lusternik and Schnirelmann's findings were that it was possible to obtain bounds using purely topological information about M.

Now, while analysist continued to pursue Lusternik and Schnirelmann's ideas in the context of smooth geometry, their ideas also attracted the attention of topologists. It was found that not only did Lusternik and Schnirelmann's invariant yield interesting information in the more general topological setting, but it in fact encoded much interesting homotopy-theoretic information whose connection with the original geometric picture remains somewhat obscure.

Over the years the theory had famous proponents such as I. James, and contributors such as G. Whitehead [5] and T. Ganea [1] who found increasingly conceptual ways of reformulating the basic ideas. Ganea, especially, became famous for making a particularly innocent looking conjecture, whose disproof was not supplied until much later, by N. Iwase [2].

What follows is an introduction to the basic theory. We will revisit the ideas at a later point, and try to understand just why they are so homotopically interesting.

Definition 1 *A covering $\mathcal{U} = \{A_i\}_{i\in\mathcal{I}}$ of a space X by (arbitrary) subspaces $A_i \subseteq X$ is said to be **categorical** if each inclusion $A_i \hookrightarrow X$ is inessential.* $\square$

Definition 2 *The **Lusternik-Schnirelmann category** of a space X, denoted $cat(X)$, is one less than the least cardinality of an open categorical cover of X. If no such integer exists we understand $cat(X) = \infty$.* $\square$

We also refer to Lusternik-Schnirelmann category as **LS category**, or most frequently simply as **category**. Unraveling the definition we see that $cat(X) \leq n$ if there exist open sets $U_1, \dots, U_{n+1} \subseteq X$ such that *i*) $\bigcup U_i = X$, and *ii*) each U_i is contractible to a point in X. While it can often be easy to produce such an inequality, showing that it is strict can be much more difficult.

It seems clear that $cat(X)$ is an invariant of the homeomorphism type of X, but maybe less clear that it is an invariant of the homotopy type of X. That it is is something you will prove shortly.

Notice that X need not connected to have finite category. Also, if $U \subseteq X$ belongs to a categorical cover of X, then U itself may have several components - as long as they all lie within the same path component of X.

Exercise 1.1 *Show that $cat(X) = 0$ if and only if X is contractible.* $\square$

Exercise 1.2 *Let a space be the union of two open subsets X, Y. Show in this case that*

$$cat(X \cup Y) \leq cat(X) + cat(Y) + 1. \qquad \square \tag{1.1}$$

Given a space X, define the (unreduced) **suspension** $\widetilde{\Sigma} X$ by means of the pushout diagram

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "j_X"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{C}X \arrow[d] \\ \widetilde{C}X \arrow[r] & \widetilde{\Sigma}X. \end{tikzcd} \tag{1.2}$$

By reindexing the intervals in the cones we get a preferred model for $\widetilde{\Sigma}X$ as a quotient of $X \times I$.

Exercise 1.3 *Show that $cat(\widetilde{\Sigma}X) \leq 1$ for any space X. Can the inequality by strict? Show that $cat(S^n) = 1$ exactly for $n \geq 1$.*

Fix a commutative ring R and let $H^*(-) = H^*(-; R)$ be your favourite ordinary cohomology theory with products and coefficient ring R. You assume H^* is singular cohomology if you wish.

Definition 3 *The R-**cup length** of a space X, denoted $cup_R(X)$, is the least integer k such that all $(k+1)$-fold cup products vanish in the reduced cohomology $\widetilde{H}^*(X; R)$. If no such integer exists we understand $cup_R(X) = \infty$.* $\square$

Clearly $cup_R(X)$ is an invariant of both the homeomorphism and homtopy type of X. Its utility lies in the following inequality, which is a fundamental bound for category.

Exercise 1.4 *Show that the inequality*

$$cup_R(X) \leq cat(X) \tag{1.3}$$

holds for any space X and any ring R.

Exercise 1.5 *Show that*

$$n \leq cat(\mathbb{R}P^n), \qquad\qquad n \leq cat(\mathbb{C}P^n) \tag{1.4}$$

where $\mathbb{R}P^n$ is real projective n-space and $\mathbb{C}P^n$ is complex projective n-space. Let $T^n = \prod^n S^1$ be the n-torus. Show that

$$n \leq cat(T^n). \qquad \square \tag{1.5}$$

How close is the bound (1.3)? It is often exact (consider S^n, for instance). However, it can also be the case that it provides little to no information.

Exercise 1.6 *Show that the difference between $cat(X)$ and $cup_R(X)$ can be arbitrarily large (Hint: Use Exercise 1.5. Choose your rings carefully.).* $\square$

We could imagine improving (1.3) by perhaps defining the *cup length* of X to be the supremum of all the various $cup_R(X)$ as we let R run over all rings. This is fruitful, and certainly provides a better approximation to $cat(X)$ than any given $cup_R(X)$, but there are still spaces with arbitrarily large category and no non-trivial cup products for any ring R.

Let us consider, then, if it might be possible to bound category from the other direction. Recall the result of Exercise 1.2.

Exercise 1.7 *Show that if X is obtained from a path connected space A by attaching an n-cell*[1]*, then $cat(X) \leq cat(A) + 1$. Show more generally, that if X is obtained from a path connected space A by attaching any number of n-cells*[2]*, then $cat(X) \leq cat(A) + 1$.* $\square$

Exercise 1.8 *Assume that X is a connected CW complex*[3]*. Show that*

$$cat(X) \leq \dim X. \qquad \square \tag{1.6}$$

Exercise 1.9 *Compute $cat(\mathbb{R}P^n)$ and $cat(T^n)$. Can you compute $cat(\mathbb{C}P^n)$?* $\square$

Note that Exercise 1.7 and its corollaries have obvious generalisations. For instance we could allow cells of varying dimensions to be attached simultaneously, or perhaps allow for more general spaces than cells to be attached. We will see such an idea crop up at a late time, but we won't need it here.

Next we would like to address the homotopy invariance of the LS category. We will first show that if X homotopy retracts off a space Y, then the category of X is necessarily less than that of Y.

Exercise 1.10 *Let $\mathcal{U}$ be an open categorical cover of a space Y and let $f : X \to Y$ be a map. Assume that f has a left homotopy inverse g. Show that*

$$f^{-1}\mathcal{U} = \{f^{-1}(U) \mid U \in \mathcal{U}\} \tag{1.7}$$

is a categorical cover of X. Conclude that $cat(X) \leq cat(Y)$. $\square$

Exercise 1.11 *Show that if $X \simeq Y$, then $cat(X) = cat(Y)$.* $\square$

This is as far as we will consider LS category at this time. You have shown in Exercise 1.11 that $cat(X)$ is an invariant of the homotopy type of X. With this in mind, it may seem startling that its origins lie in smooth geometry, and in the work of Lusternik-Schnirelmann which led to the following theorem [3] (although see Palais [4] for the definitive statement).

[1] i.e. X is the pushout of a span $A \xleftarrow{\varphi} S^{n-1} \hookrightarrow D^n$ for some map φ.

[2] i.e. X is the pushout of a span $A \xleftarrow{\varphi} \coprod S^{n-1} \hookrightarrow \coprod D^n$ for some map φ.

[3] You may assume that $X_0 = *$.

Theorem 1.1 (Lusternik-Schnirelmann Theorem) *Let M be a smooth compact manifold and denote by $crit(M)$ the minimum number of critical points of any smooth function $f : M \to \mathbb{R}$*[4]*. Then*

$$cat(M) + 1 \leq crit(M). \quad \blacksquare \tag{1.8}$$

References

[1] T. Ganea, *Lusternik-Schnirelmann Category and Strong Category*, Illinois J. Math. **11** (1967), 417-427.

[2] N. Iwase, *Ganea's Conjecture on Lusternik-Schnirelmann Category*, Bull. Lond. Math. Soc. **30** (1998), 623-634.

[3] L. Lusternik, L. Schnirelmann, *Methodes Topologiques dans les Problemes Variationnels*, Hermann, Paris, (1934).

[4] R. Palais, *Lusternik-Schnirelmann Theory on Banach Manifolds*, Topology **5** (1966), 115-132.

[5] G. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, (1978).

[4] A critical point of a function $f : M \to \mathbb{R}$ is a point $p \in M$ at which the derivative, or tangent map, of f vanishes.